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PERTURBED MOTION OF A VISCOUS FLUID LAYER ON THE SURFACE OF A ROTATING CYLINDER

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We investigate the plane flow of a viscous fluid layer plane flow on the surface of a cylinder rotating with a constant angular velocity with account for inertia forces and the acceleration of gravity.

In [1, 2] equations were obtained for the evolution of a thin fluid layer on a rather slowly rotating cylinder without account for inertia forces. In [1, 3] experimental investigations were carried out. In [2] a theorem on the existence and uniqueness of a stationary solution was proved for the case of a small gravity effect. In [4] a qualitative investigation of the forms of equilibrium of plane layers was carried out without account for gravity forces.

Below, we apply a direct method for solving a nonstationary problem. We obtain and analyze results of calculations for flows with and without the effect of gravity. In the process of evolution, periodic disturbances develop on a free surface in a gravity field, the number of local extrema and the maximum value of the free surface radius grow, the uniformity of their distribution over the cylinder circumference is disrupted, individual maxima increase up to values at which the process of the computations is terminated. The form of the free surface and the intervals of the values of the angle within which the maximally growing disturbances are localized agree with [1, 3]. An analysis of the numerical solution of the equations in [1, 2] showed that in those works the researchers did not observe the development of wave disturbances over the cylinder circumference due to neglect of nonlinear terms.

1. It is convenient to consider a plane flow of a viscous fluid in a relative coordinate system O, η, φ fixed in the cylinder. The Navier-Stokes equations are augmented with an equation for the unknown surface, boundary conditions of absence of slip at the cylinder surface $\eta = 1$, absence of viscous interaction with the ambient medium, and continuity of normal stresses on the free surface $\eta = h(\varphi, \tau)$, the condition of flow periodicity in the angular coordinate, and initial conditions. The initial-boundary value problem involves three dimensionless parameters: the Reynolds, Froude, and Weber numbers

$$\mathrm{Re} = \frac{\rho R_0^2 \omega_0}{\mu}, \quad \mathrm{Fr} = \frac{R_0 \omega_0^2}{g}, \quad \mathrm{We} = \frac{\rho R_0^3 \omega_0^2}{\sigma},$$

where R_0 is the cylinder radius; ω_0 is the angular velocity of the cylinder; ρ, μ, and σ are the density and coefficients of dynamic viscosity and the surface tension of the fluid; g is the acceleration of gravity. In the case of a fast rotation Re >> 1, Fr >> 1, and We >> 1. In this case the relative change in the fluid flow in the transverse direction is much smaller than in the radial one, the radial velocity component is much smaller than the transverse velocity component, and the relative thickness of the layer is much smaller than the disturbance wavelength. This makes it possible to obtain equations and boundary conditions in a first approximation:

$$\eta^2 (\omega_\tau + v\omega_\eta + \omega\omega_\varphi) + 2\eta v (1 + \omega) =$$

$$= - p_\varphi + \frac{\eta}{\mathrm{Re}} (3\omega_\eta + \eta\omega_{\eta\eta}) - \frac{\eta \cos (\varphi + \tau)}{\mathrm{Fr}}; \tag{1}$$

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$$p_\eta = (1+\omega)^2\eta\,; \tag{2}$$

$$(\eta v)_\eta + (\eta\omega)_\varphi = 0\,; \tag{3}$$

$$\eta = h(\varphi, \tau), \quad h_\tau + \omega h_\varphi = v\,; \tag{4}$$

$$\eta = 1, \quad v = 0\,; \tag{5}$$

$$\eta = 1, \quad \omega = 0\,; \tag{6}$$

$$\eta = h(\varphi, \tau), \quad \frac{1}{\mathrm{We}}\left(\frac{1}{h} - \frac{h_{\varphi\varphi}}{h^2}\right) = p - p_a\,; \tag{7}$$

$$\eta = h(\varphi, \tau), \quad \omega_\eta = 0\,; \tag{8}$$

$$\begin{gathered} h(\varphi, \tau) = h(\varphi + 2\pi, \tau), \quad \omega(\varphi, \tau) = \omega(\varphi + 2\pi, \tau), \\ v(\varphi, \tau) = v(\varphi + 2\pi, \tau), \quad p(\varphi, \tau) = p(\varphi + 2\pi, \tau)\,; \end{gathered} \tag{9}$$

$$\tau = 0, \quad h = h_0(\varphi), \quad \omega = \omega_0(\varphi), \quad v = v_0(\varphi)\,. \tag{10}$$

Here $\omega = w/\eta - 1$ is the relative angular velocity; v and w are the radial and peripheral velocities; p and p_a are the pressures in the layer and in the undisturbed surrounding medium.

2. Let us make use of one step of the direct method. Integration of Eqs. (1)–(3) over η from $\eta = 1$ to $\eta = h(\varphi, \tau)$ yields the relations

$$\begin{gathered} \frac{\partial}{\partial\tau}\int_1^h \eta^2\omega d\eta + \frac{\partial}{\partial\varphi}\int_1^h \eta^2\omega^2 d\eta + 2\int_1^h \eta v d\eta + \int_1^h \eta v\omega d\eta = p(h, \varphi, \tau)\frac{\partial h}{\partial\varphi} - \\ - \frac{\partial}{\partial\varphi}\int_1^h p d\eta - \frac{(h^2-1)}{2\,\mathrm{Fr}}\cos(\varphi + \tau) + \frac{1}{\mathrm{Re}}\left(h\omega\,|_{\eta=h} - \partial\omega/\partial\eta\,|_{\eta=1} - \int_1^h \omega d\eta\right); \end{gathered} \tag{11}$$

$$p = p(h, \varphi, \tau) - \int_\eta^h \eta(1+\omega)^2 d\eta\,; \tag{12}$$

$$\eta v = -\int_1^h (\eta\omega)_\varphi d\eta\,. \tag{13}$$

With the aid of substitution of variables, the flow region is transformed into a circle

$$\zeta = \frac{\eta - 1}{\delta(\varphi, \tau)}, \quad 0 \le \zeta \le 1, \quad \delta(\varphi, \tau) = h(\varphi, \tau) - 1\,, \tag{14}$$

where $\zeta = 0$ corresponds to the cylinder surface and $\zeta = 1$ to the free surface of the layer. Let us assume that the dependence of the relative angular velocity ω on the variable ζ, which satisfies conditions (6) and (8), has the form

$$\omega(\zeta, \varphi, \tau) = -T(\varphi, \tau)\zeta\left(1 - \frac{1}{2}\zeta\right), \tag{15}$$

where $T(\varphi, \tau)$ is a function that is periodic in φ and that requires determination. Substitution of Eq. (15) into Eq. (13) permits one to obtain the formula

$$\eta v = \sum_{n=2}^{4} A_n(T, \delta, T_\varphi, \delta_\varphi)\zeta^n, \tag{16}$$

$$A_2 = (-T\delta_\varphi + \delta T_\varphi)/2, \quad A_3 = ((1-\delta)T\delta_\varphi + \delta(2\delta - 1)T_\varphi/2)/3,$$

$$A_4 = (\delta T\delta_\varphi - \delta^2 T_\varphi/2)/4.$$

From relation (12) we can derive a formula for the pressure distribution in the layer:

$$p(\zeta, \varphi, \tau) = p(h, \varphi, \tau) + \sum_{n=0}^{6} b_n\zeta^n, \tag{17}$$

$$b_0 = -\delta - \frac{\delta^2}{2} + \frac{T\delta}{24}(16 + 5\delta) - \frac{T^2\delta}{120}(16 + 11\delta), \quad b_1 = \delta,$$

$$b_2 = \frac{\delta^2}{2} - T\delta, \quad b_3 = \frac{T\delta}{3}(1 - 2\delta) + \frac{T^2\delta}{3}, \quad b_4 = (T\delta^2 - T^2\delta(1-\delta))/4,$$

$$b_5 = T^2\delta(1 - 4\delta)/20, \quad b_6 = T^2\delta^2/24.$$

Integral relation (11) with allowance for Eq. (17) is transformed to

$$T_\tau = U\delta_\varphi + VT_\varphi - \frac{60}{\mathrm{We}(1+\delta)^2 E_0(\delta)}\left(\delta_\varphi + \delta_{\varphi\varphi\varphi} - \frac{2\delta_\varphi\delta_{\varphi\varphi}}{1+\delta}\right) +$$

$$+ \frac{30\cos(\varphi + \tau)}{\mathrm{Fr}}\,\frac{(2+\delta)}{E_0(\delta)} - \frac{10T}{\mathrm{Re}\,\delta^2 E_0(\delta)}(6 - 3\delta - \delta^2), \tag{18}$$

where

$$E_0(\delta) = 20 + 25\delta + 9\delta^2; \quad U(\delta, T) = T^2U_2(\delta) + TU_1(\delta) + U_0(\delta);$$

$$V(\delta, T) = TV_1(\delta) + V_0(\delta); \quad U_2(\delta) = \frac{1}{\delta E_0(\delta)}\left[\frac{1}{42}(336 + 553\delta + 38\delta^2) - \right.$$

$$\left. - \frac{(4+5\delta)}{12(1+\delta)}(20 + 50\delta + 27\delta^2)\right]; \quad U_1(\delta) = (40 + 50\delta)/E_0(\delta);$$

$$U_0(\delta) = -60(1+\delta)/E_0(\delta); \quad V_1(\delta) = \frac{1}{E_0(\delta)}\left[\frac{1}{21}(336 - 49\delta - 34\delta^2) - \right.$$

$$\left. - \frac{(8+5\delta)}{24(1+\delta)}(20 + 50\delta + 27\delta^2)\right]; \quad V_0(\delta) = \delta(50 + 25\delta)/E_0(\delta).$$

The equation for determining the unknown free surface has the form

TABLE 1. Dependence of the Characteristics of Layer Disintegration on the Parameters Re, Fr, and We

ω_0, rps	Re	Pr	We	τ_p	n_{max}	θ_p	D_θ
5	17.7	2.5	277.6	31.53	16	2.80	2.0–2.9
6	21.2	3.6	399.7	20.66	17	3.37	2.9–4.0
9	31.8	8.2	899.4	8.52	22	5.48	5.2–5.7
15	53.0	22.6	2498.0	3.53	33	5.87	2.9–3.5; 5.2–5.5
24	84.0	58.0	6396.0	2.17	38	5.21	1.4–1.8; 4.9–5.4
48	169.7	231.8	25580.0	1.93	44	2.00	1.6–2;.1; 4.6–5.1

$$\delta_\tau = H(\delta)\, T_\varphi + R(\delta, T)\, \delta_\varphi , \tag{19}$$

$$H(\delta) = \frac{\delta(8 + 5\delta)}{24(1 + \delta)}, \quad R(\delta, T) = \frac{T(4 + 5\delta)}{12(1 + \delta)} .$$

System of equations (18) and (19) is augmented with the condition of periodicity in the angular coordinate and also with periodic initial conditions:

$$\delta(\varphi, \tau) = \delta(\varphi + 2\pi, \tau), \quad T(\varphi, \tau) = T(\varphi + 2\pi, \tau); \tag{20}$$

$$\delta(\varphi, 0) = \delta_0(\varphi), \quad T(\varphi, 0) = T_0(\varphi). \tag{21}$$

3. The solution of initial-boundary value problem (18)–(21) is sought by the method of straight lines in the region $0 \le \varphi \le 2\pi$, $\tau > 0$. The flow region is subdivided by N rays: $\varphi = \varphi_n = n\Delta\varphi$, $n = 1, 2, \dots, N - 1$ ($\Delta\varphi = 2\pi/N$). Derivatives with respect to φ on reference rays are represented by finite-difference relations, and Eqs. (18) and (19) are reduced to a system of ordinary differential equations:

$$\frac{dT_n}{d\tau} = f_1\,[\tau, \delta_n, (\delta_\varphi)_n, (\delta_{\varphi\varphi})_n, (\delta_{\varphi\varphi\varphi})_n, T_n, (T_\varphi)_n], \tag{22}$$

$$\frac{d\delta_n}{d\tau} = f_2\,[\delta_n, (\delta_\varphi)_n, T_n, (T_\varphi)_n], \tag{23}$$

where the functions f_1 and f_2 represent the right-hand sides of Eqs. (18) and (19) after discretization over φ. Conditions (20) and (21) require the compliance with the equalities

$$\delta_n(\tau) = \delta_{N+n}(\tau), \quad T_n(\tau) = T_{N+n}(\tau); \tag{24}$$

$$\delta_n(0) = \delta_n^0, \quad T_n(0) = T_n^0. \tag{25}$$

System of ordinary differential equations (22) and (23) with auxiliary conditions (24) and (25) is integrated by the Runge-Kutta method with a constant step using formulas of the fourth order of accuracy. The value of N was varied and amounted to 180, 360, 720, the integration step in time was varied from $\pi/2000$ to $\pi/200$. The accuracy of the computations was controlled by the condition of fluid mass conservation in the layer:

$$M = \frac{1}{2}\int_0^{2\pi} [h^2(\varphi, \tau) - 1]\, d\varphi .$$

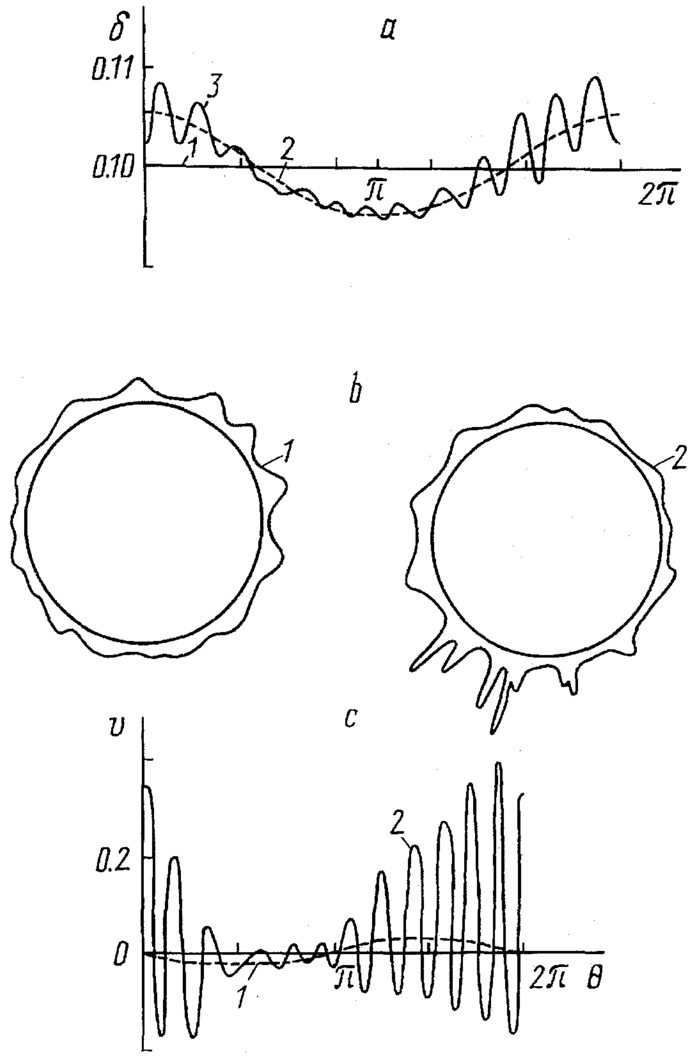


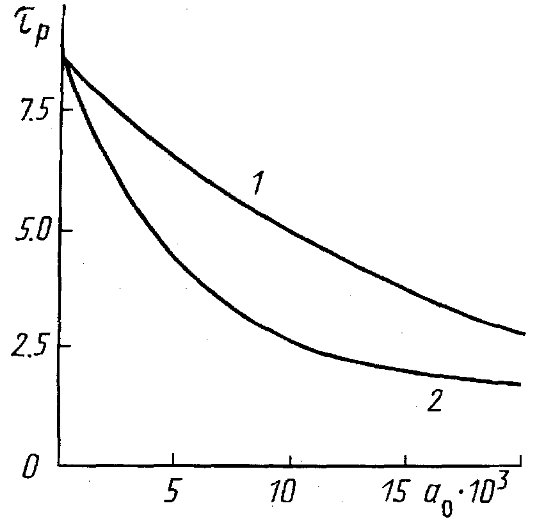
Fig. 1. Shape of a free surface at different instants of time.

Computations were terminated when the absolute maximum of the layer thickness attained five maximum values of it at the initial instant of time; the corresponding time value is denoted everywhere below by τ_p.

4. The numerical solution of the problem was carried out under conditions corresponding to the experiments in [3]: the fluids used were aqueous solutions of glycerin at 20°C; the cylinder radius was equal to 1.23, 2.5, and 3.5 cm; angular velocities ranged from 3 to 200 rps.

Suppose that at the initial instant of time the layer has a constant thickness δ^0 and the cylinder moves as an entity. Due to the instability, one maximum and one minimum appear on the free surface. This is followed by the development of other small-amplitude disturbances distributed uniformly over the cylinder surface. At the initial stage of evolution the number of local extrema and the maximum value of the free surface radius increase, then the uniformity of their distribution is disrupted, and individual maxima grow further. The rearrangement of the flow occurs. Figure 1a and b shows the shape of the free surface of the layer in a fixed coordinate system $r = \eta$, $\theta = \varphi - \tau$ at $\delta^0 = 0.10$, Re = 17.7, Fr = 2.52, We = 277.7; in Fig. 1a line 1 corresponds to $\tau = 0$, line 2 to $\tau = \pi$, and line 3 to $\tau = 7\pi$; in Fig. 1b: 1) $\tau = 9\pi$; 2) $\tau = \tau_p = 31.53$. The condition for the termination of computations is determined by the inequality $\delta_{\max} \geq 5\delta^0$.

In Fig. 1c curves for $v(\tau, \theta)$ are given: 1) $\tau = 2\pi$; 2) $\tau = 9\pi$. In Table 1 the values of τ_p, the maximum number of disturbances $n_{\max}$ developed at time τ_p, the angle θ_p corresponding to the largest value of the free surface radius at time τ_p, and the intervals of θ values with the maximally growing disturbances are presented as functions of the parameters Re, Fr, and We. Calculations demonstrated the existence of two subintervals: $D_\theta' = \{5\pi/8 < \theta < 5\pi/4\}$ and $D_\theta'' = \{3\pi/2 < \theta < 2\pi\}$, in full conformity with the experiment of [3].



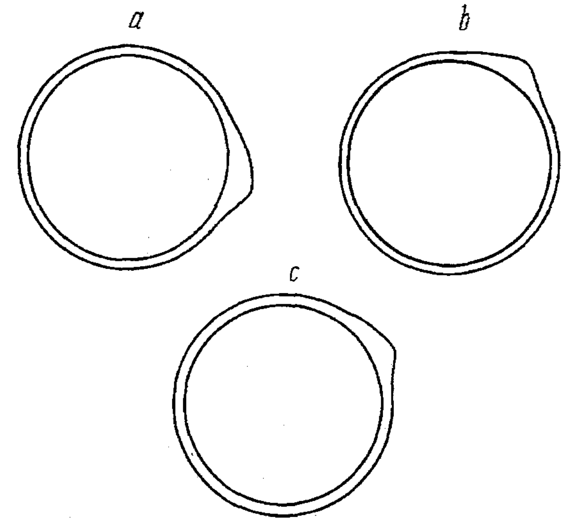


Fig. 2. Dependence of τ_p on the form and amplitude of initial disturbances at Re = 31.8, Fr = 8.2, We = 899.4. 1) $k = 4$; 2) 8.

Fig. 3. Bulges on a free surface at Re = 11.6, Fr = 0.40, We = 44.4. a) $\tau = \pi$; b) $3\pi/2$; c) $7\pi/2$.

If the initial profile of the free surface of the layer rotating as a solid body is specified in the form

$$\delta_0(\varphi) = 0.1 + a_0 \sin k\varphi\,, \quad T_0(\varphi) = 0\,, \quad k = 1\,,\, 2\,,\, \ldots\,,$$

and the dimensionless parameters of the problem are equal to Re = 31.8, Fr = 8.15, and We = 899.4, then initial disturbances develop in the first stage. Next, new disturbances of smaller amplitude appear between the points of a maximum on the free surface. The amplitude of secondary disturbances remains small, the instability in the layer develops due to the growth of one of the main maxima on the free surface according to the above-outlined scenario. In Fig. 2 the dependence of the time τ_p on the amplitude of initial disturbances a_0 is given for $k = 4$ (curve 1) and $k = 8$ (curve 2).

Figure 3 demonstrates the layer at three instants of time with an initial disturbance of the form

$$\delta_0(\varphi) = 0.1 - 0.05 \sin\varphi\,, \quad T_0(\varphi) = 0\,.$$

The dimensionless parameters are Re = 11.6, Fr = 0.40, and We = 44.4. On the free surface there are burrs or bulges recorded experimentally [1, 3].

In the case where the acceleration of gravity is ignored, the initial layer of constant thickness is preserved. A disturbed layer disintegrates due to the growth of the initial disturbances, with the uniformity in the location of the local extrema of both the basic and secondary disturbances being preserved up to the instant of termination of the computations τ_p.

Conclusion. Account for the nonlinear interaction of disturbances makes it possible to refine the mechanism underlying the disintegration of a fluid layer on the surface of a rotating cylinder.

NOTATION

O, r, θ, fixed cylindrical coordinate system; O, φ, φ, relative coordinate system fixed in the cylinder; $f_\tau = \partial f/\partial\tau$, $f_\eta = \partial f/\partial\eta$, $f_\varphi = \partial f/\partial\varphi$, $f_{\varphi\varphi} = \partial^2 f/\partial\varphi^2$, $f_{\varphi\varphi\varphi} = \partial^3 f/\partial\varphi^3$, derivatives; τ_p, dimensionless time in which the absolute maximum of the layer thickness attains five maximum values of it at the initial instant of time (the instant of termination of the computations).

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